Leverage Cycles in the Household Sector
Assessing the Early Warning Signs in Canada and the United States

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Abstract

The leverage cycles of households in Canada and the United States are compared to the the optimal debt/net worth ratio prescribed by a stochastic optimal control (SOC) model. The SOC model employed extends Jerome Stein's 2003 optimal leverage metric by modeling speculative capitals gains as a geometric Brownian motion with a variable trend rate. The difference between the normalized observed and optimal leverage is an early warning sign (EWS) of financial fragility. We find that Canadian households are in a fragile state. Therefore, a negative shock to the Canadian real estate sector could ignite a crisis but this is unlikely to be as severe as the US housing market crash in 2008/09.

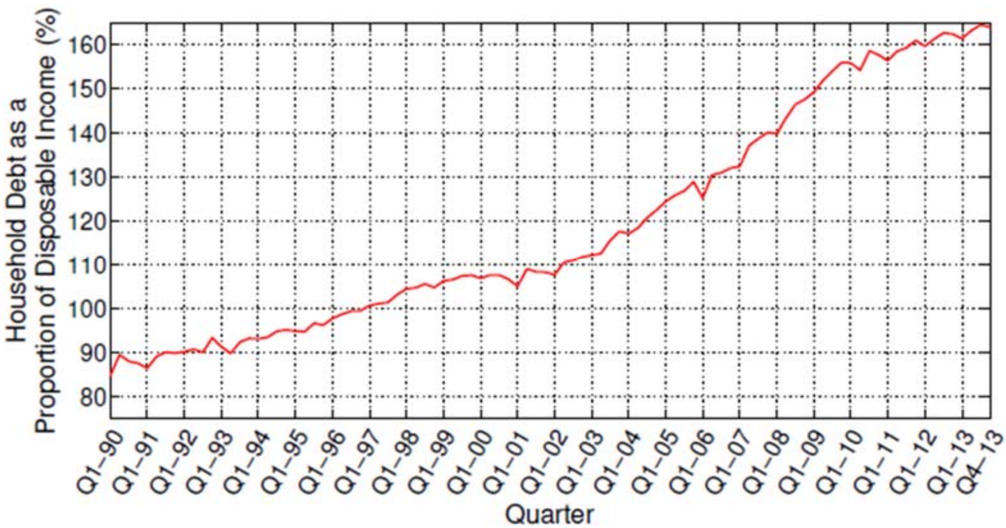
I. Introduction

In spite of its extensive economic ties with the United States, the Canadian economy weathered the 2008 financial crisis better than any other G7 country (IMF, 2013). In particular, the resilience of Canadian credit markets has – to date – been impressive. However, there is increasing concern about the sustainability of Canada's housing price bonanza (The Globe and Mail, 2013; The Economist, 2013a, 2013b; Financial Times, 2014). Housing price gains in Canada's largest real estate markets have developed in tandem with a rapid build up of household debt – reaching 164% of disposable income in the third quarter of 2013 (see Fig. 1), on par with the levels seen in the US in 2006 and 2007. Is this level of household debt sustainable? Does the increase in Canadian households' assets justify higher debt-to-net worth ratios? We apply and extend Jerome Stein's model of optimal leverage to answer in the negative. Since the collapse of Bear Stearns in March 2008, Canadian household leverage has remained precariously above our measure of optimal leverage. Indeed, since mid-2011 the difference between observed and optimal leverage has been greater than 2 standard deviations above the sample mean, and growing. One way or another the gap between households' actual and optimal leverage ratio will close.

The stochastic optimal control (SOC) model developed in section 3 allows for direct comparison of the optimal and actual household leverage. A positive difference between these two figures is an 'early warning sign' (EWS) indicating financial fragility and the potential for a crisis. Section 4 applies this model retrospectively to the 2007/08 US real estate boom-bust cycle and to the current Canadian market.

We find that the Canadian household sector has had a significant early warning sign since 2009 (through 2013 Q4), indicating a high degree of financial fragility. Looking beyond this important conclusion we explore the role of bank lending (or, rather, its sudden stop) in igniting the US household sector's debt crisis. Section 6 reviews the findings and suggests avenues for future research.

Figure 1: Ratio of Canadian Households' Debt to Disposable Income



Source: Quarterly data. Ratio of total household sector stock of debt to total gross disposable income. Statistics Canada, National Economic Accounts (NEA), tables 378-0121 and 378-0037.

II. Manias Metrics and Balances

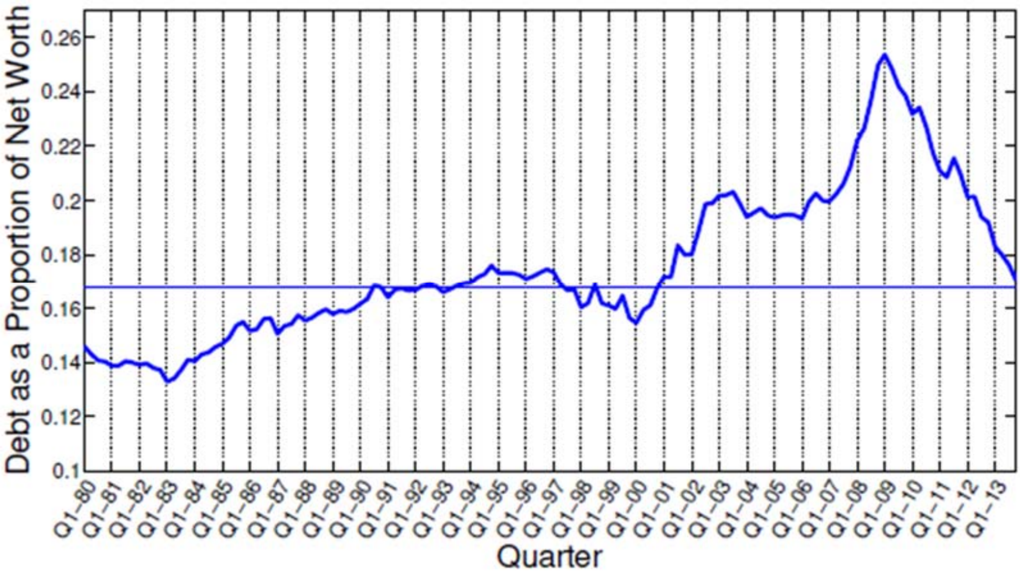
During the 'Great Moderation', from the mid-1980s to the mid-2000s, monetary policy was thought to have effectively insulated the real economy from the occasional financial boom-bust cycle. It was not until the full scale of the financial crisis emerged in autumn 2008 that most economists began to heed Minsky's (1993, p. 6) plea to "take banking seriously as a profit-seeking activity." Although we do not here present a formal Minskyan model, we follow an emerging macro-financial literature that, at its core, looks to Hyman Minsky's Financial Instability Hypothesis (FIH) as a motivational framework. Of course, Minsky's analysis was not restricted to formally incorporated 'banks'; rather the term is shorthand for financial investment units, a category which includes shadow banking, financial arms of non-financial firms and so on (Minsky, 2008[1986]). The present model considers households as profit-seeking investment units that are able to obtain credit lines (primarily, mortgages) with which to invest (primarily in real estate). Further, the model incorporates two other key Minskyan insights: (i) that risk builds up during expansionary phases, and; (ii) financial market participants act on capital gains signals, even if they cannot be related to "fundamentals".

There is a large body of literature focusing on the nature of financial boom-bust cycles (e.g., Stiglitz, 1990; Fama, 1998; Geanakoplos, 1997; LeRoy, 2004; Semmler & Mittnik, 2013). Earlier attempts to model the patterns within standard macro- economic frameworks have been unable to capture the observed dynamics in financial markets (e.g., Kiyotaki & Moore, 1997; Holmström & Tirole, 1997; Bernanke, Gertler, & Gilchrist, 1999). This has led to the rise of non-standard approaches including agent-based models (e.g., Gallegati, Palestrini, & Rosser, 2011; Theobald, 2012) and stochastic equilibria models (e.g., He & Krishnamurthy, 2008; Semmler & Bernard, 2012; Brunnermeier & Sannikov, 2014). Yet, in virtually all cases, the object of study is the active financial participant, such as banks and hedge funds. To the extent that the household sector is included, it tends to be a passive supplier of credit to financial experts. Jerome Stein's optimal leverage model is an exception to this rule as it can – with appropriate adjustments – be applied to the asset returns and balance sheet position of any market sector or group (see, Stein, 2003; Fleming & Stein, 2004; Stein, 2006, 2011, 2012). This flexibility stems from Stein's goal of estimating a sector's optimal leverage based on total asset returns and debt servicing costs. That is, the objective of the Stein model is not to simulate leveraging behavior over the credit cycle, but to determine the sustainability of observed leverage.

IIA. Stein's Early Warning Sign for Crises

Clearly, households' adjust their portfolio far less frequently than financial firms. Nevertheless, the household sector has been an active participant in financial cycles. For example, during the recent housing price boom in the United States, household leverage (= debt/net worth) followed a procyclical pattern, jumping from an average of 16% in the 1990s to 20% in the early 2000s – well above historical norms (see Fig. 2). As the housing crisis took hold, the sector's leverage peaked at over 25% in early 2009. This implies a passive management of portfolios. From mid-2009 onwards this passivity gave way as households began to pay down their liabilities and housing prices stabilized. This pattern of recovery is suggestive of a (lagged) procyclical leverage cycle. There are many ways one might simulate leverage over this boom-bust cycle, but Stein's focus is on a more practical tool: an early warning sign measure of financial fragility. Such a model is especially important given that, in every asset-price boom, participants defend the capital gains as justified by "the fundamentals."

Figure 2: Ratio of US Households' Leverage (Debt to Net Worth)



Quarterly data, period ending. Ratio of total liabilities to net worth for the household and non-profit organizations serving household sector. Solid line is the pre-crisis mean leverage ratio (from 1980 Q1 to 2008 Q1). Federal Reserve Economic Database (FRED), series TNWBSHNO and TLBSHNO.

In the recent cycle and during the 1980s US real estate boom, Stein (2003, 2012, chapter 5) shows that households' observed leverage, f_t , was far above the optimal leverage, f_t^*, computed in his model. Stein defines this difference as

$$\psi_t \equiv f_t - f_t^*$$

which he calls an early warning sign (EWS) when $\Psi_t >> 0$. The broad applicability and recurrence of the EWS prior to many financial crises (see, Stein, 2006, 2012) makes this an attractive approach. However, in each empirical application, Stein analyzes only completed boom-bust cycles.

We break from this practice to analyze the current Canadian housing market and current Canadian household balance sheet data.[1] Canadian housing prices have grown rapidly over the past several years, suffering only a minor retrenchment in the depths of the 2008 financial crisis south of the border. We find that, indeed, the EWS ($\psi \gg 0$) is well above historical norms in Canada (section 5). However, the EWS metric indicates only that financial fragility exists; it does not allow us predict if, let alone when, a housing crisis might occur. That is, Stein's approach does not include a mechanism – conceptual or empirical – for the onset of a debt crisis. Therefore, evidence of financial fragility in Canada offers a cautionary warning to policymakers that must be interpreted within the particular economic and regulatory context.

III. The Stochastic Control Model of Optimal Leverage

IIIA. Set Up

The model developed here is an extension of the optimal household leverage model in Stein (2003). Stein has refined this model in many subsequent vintages (e.g., Fleming & Stein, 2004; Stein, 2006, 2011, 2012), however the 2003 version has some distinct advantages. In more recent vintages, agents optimize a concave function of terminal net worth by their choice of leverage. Stein (2003) employs a more standard set up of an infinitely-lived agent optimizing consumption over time, which we adopt here. Second, the agent chooses both leverage and consumption. Finally, in Stein's older model, capital productivity, speculative price gains and the interest rate are random variables. Later versions treat one of these (typically productivity) as a constant or deterministic function. Although it adds some empirical complexities, the added realism from including all three rates of return as random variables is a welcomed extension.

The agent's objective function is

$$max E_0\left[\int_0^\infty e^{\delta t} U(C_t) dt\right] \tag{1}$$

where $0 < \delta < 1$ is the pure rate of time preference and U (•) is a well-behaved, twice-differentiable function of the hyperbolic absolute risk aversion (HARA) type. The choice variables c_t , f_t are consumption and debt normalized by net worth, respectively. The problem is solved subject to the stochastic differential equation (12), which represents the evolution of net worth X_t . Importantly, the solution methodology employed maintains the first two moments of the

[1] At the time of this writing the latest complete set of data runs through 2013 Q4.

stochastic processes in (12) – a crucial difference from typical dynamic optimization models that maintain only the first.

Although equation (1) appears as a typical rational expectations optimization, the constraint is a stochastic differential equation (SDE). An analytical solution, therefore, relies on stochastic optimal control (SOC) theory (see Fleming & Rishel, 1975). The solution methodology is continuous-time dynamic programming (DP). This means that the recursive Hamilton-Jacobi-Bellman (HJB) equation is defined over infinitesimal time steps dt, which requires the use of a backward operator, $A\mathbf{u}$, that limits the centrality of the expectations operator (Chang, 2004, chapter 4).[2] In the more common forward-looking, discrete-time models the expectations operator is applied directly to the period-ahead value function thereby generating a certainty-equivalent DP equation in which all random variables are set to their mean value, typically zero. Dynamic programming in the SOC setting also leads to a deterministic equation (a non-linear 2nd-order differential one), but it is not certainty-equivalent. In SOC the HJB equation retains the (squared) diffusion parameters (see Appendix A). Moreover, as we show below, the optimal control variable that we seek, f_t^*, fluctuates over the particular realization path of the state variable, net worth Xt. Although the SOC solution implies a steady-state Markov equilibrium, this solution is incidental to our focus on households' optimal leverage ratio that is implied by the contemporaneous data.

The agent's consumption comes entirely from its current net worth, Xt. Ideally, one would like to include labor-market income which could add to net worth or to consumption. However, to maintain a tractable analytical solution, we do not incorporate such an extension. This choice is further justified by the fact that the present focus is on household balance sheet management to which the narrower portfolio approach is more appropriately honed. Therefore, in the present model, the evolution of consumption is always limited by the evolution of the state variable, *i.e.*, net worth.

The specific time path of net worth can be modeled by many different specifications, but the basic framework must begin with the balance sheet identity

[2] The backward operator is a second order partial differential operator.

$$K_t = L_t + X_t \; for \; t \geq 0 \tag{2}$$

where Kt denotes total assets (capital), Lt is the agent's stock of debt and Xt is the book value of equity/net worth. The book value of net worth means Xt adjusts such that (2) holds identically. The balance sheet equation must also hold identically in terms of its flows,

$$dX_t = dK_t - dL_t \tag{3}$$

which is simply the total differentiation of (2). Equation (3) says the difference between the (instantaneous) change in the value of assets and the value of liabilities is equal to the change in net worth, dXt.

As stated, the agent controls the evolution of net worth by its choice of leverage,

$$f_t \equiv \frac{L_t}{X_t} \;\; which \; implies \;\; 1 + f_t = \frac{K_t}{X_t} \tag{4}$$

The leverage ratio, f_t is the key variable of the model. Our goal is to find the value f_t^* that optimizes (1) in an inherently uncertain environment. This uncertainty is modeled as a set of stochastic processes – specifically, controlled diffusions – that propel the evolution of net worth forward through time.

Stein (2012, Chapter 4) recognizes that there is no such thing as a "true" model of financial evolutions. A model, therefore, should provide a plausible set of variables and relationships with which to study actual economic phenomena. With this in mind we build on the models of Stein (2003) and Fleming and Stein (2004). Of these two papers, which lay the foundation for the authors' later work, Stein (2003) is of particular interest since it models three stochastic processes: speculative capital gains, capital productivity and the interest rate.[3] We recapitulate this model, but improve upon it by depicting speculative capital gains as geometric Brownian motion. The drift component is treated as a variable positively correlated with net worth. Thus, the expected rate of capital gains fluctuates procyclically over the the business cycle.

Keeping in mind the identities (3) and (4), we begin by defining total capital as the product of its physical stock, N_t, and its price, P_t:

[3] Since his 2003 paper, all of Stein's models in this area have used only two stochastic processes, typically treating capital productivity as a constant or as deterministically growing.

$$K_t = P_t N_t \tag{5}$$

On an analytical level, at least, this specification delineates observed changes in total assets into "real capital" changes (dN_t) and pure speculative price variation (dP_t). Throughout this paper P_t and dP_t represent the nominal price level and its change. Therefore, capital gains are always understood as purely speculative. This distinction is present in all versions of Stein's model, though often less explicitly represented than in (5).[4]

Total differentiation of (5) gives the evolution of capital as

$$dK_t = I_t dt + K_t \frac{dP_t}{P_t} \tag{6}$$

where the final equality indicates that investment, I_t directly increases the volume of capital stock at a given price and $\frac{dP_t}{P_t}$ denotes (speculative) capital gains.

Following the finance literature, capital gains are modeled as geometric Brownian motion:

$$\frac{dP_t}{P_t} = \mu_t dt + \sigma_p dW_p, \ where \ \ dW_{p,t} = \epsilon_p \sqrt{dt} \ and \, \epsilon_p \sim N(0,1) \tag{7}$$

Stein (2003) considers the capital gains drift rate to be a constant value, $\bar{\mu} \geq 0$. We loosen this assumption by modeling the drift rate as an *implicit* function of time *t*, which can become negative in a severe downturn. Formally, we consider μ_t as a positive function of the change in total assets:

$$\mu_t \equiv \mu(dK_t), \qquad \mu' > 0$$

We do not offer a precise functional form of μ_t since it is unobservable and untestable. In section 4 μ_t is operationalized as the smoothed difference between changes in capital's market value and its productivity.

The final stochastic element comes from the capital productivity variable, b_t, which identically equals

[4] Readers familiar with the Cambridge controversy will be aware of the impossibility of aggregat- ing across physical stocks of capital without reference to capital's price and, hence, capital's rate of return. However, this is not a concern for the narrow forces of the present paper: Because we are focusing on the housing market, Nt can be thought of as the number of houses or the total floor area of the stock of houses. Of course, in general, caution should be exercised on this point.

$$Y_t \equiv b_t K_t$$

where Y_t denotes output (or value-added). In keeping with Stein (2003), this is a non-differential stochastic process with mean $\beta > 0$,

$$b_t dt = \beta dt + \sigma_b dW_b, \quad where \quad dW_{r,t} = \epsilon_r \sqrt{dt} \quad and \quad \epsilon_r \sim N(0,1) \tag{8}$$

Hence, capital productivity randomy oscillates around β with a standard deviation of $\sigma_b\sqrt{dt}$.

Capital productivity process and the debt servicing evolution are connected through the macroeconomic identity for flows to and from the agent's stock of debt

$$dL_t = L_t r_t dt + C_t dt + I_t dt - Y_t dt \tag{9}$$

where the interest servicing cost $L_t r_t$ is observable (see section 4). Equation (9) says that changes in interest payments, consumption and investment spending above (below) the change in net output, $Y_t dt$, are financed by debt accumulation, $dL_t > 0$ (re- payment, $dL_t < 0$). Further, given equation (8) above, $Y_t dt$ in (9) can be replaced by

$$Y_t dt = K_t b_t dt = K_t(\beta dt + \sigma_b dW_b) \tag{10}$$

Hence

$$dL_t = L_t r_t dt + C_t dt + I_t dt - K_t(\beta dt + \sigma_b dW_b) \tag{11}$$

Finally, all three stochastic process can be brought into a single equation for the evolution of net worth. Substituting equation (7) into (6), and (11) into the identity for balance sheet flows, (3), yields

$$\begin{aligned} dX_t = X_t\{[(1 + f_t)(\beta + \mu_t) - f_t \cdot r_t - c_t]dt \\ +(1 + f_t)(\sigma_t dW_b + \sigma_p dW_p)\} \end{aligned} \tag{12}$$

All terms on the righthand side of (12) have been normalized by net worth; hence $c_t \equiv \frac{C_t}{X_t}$ and, recall, $f_t = \frac{L_t}{X_t}$ and $1 + f_t = \frac{K_t}{X_t}$. Since the agent consumes only out of net worth, it must maximize (1) subject to (12) via an optimal choice of c_t^* and f_t^* at every point in time. We derive these optima below.

IIIB. Solving the Model

For tractability the felicity function is operationalized as a homogenous power utility. This implies an agent with decreasing absolute risk aversion but constant relative risk aversion. Specifically, let

$$U(C_t) = \frac{1}{\gamma}C_t^{\gamma}, \ \ 0 < \gamma < 1 \tag{13}$$

with absolute and relative risk aversion coefficients,

$$\textit{Absolute:} -\frac{U''}{U'} = \frac{1-\gamma}{C} \qquad\qquad \textit{Relative:} -\frac{U''}{U'} \cdot C = 1 - \gamma$$

With this specification the agent maximizes the value function

$$V(X) = \max_{c_t, f_t} E_0\left[\int_0^{\infty} e^{-\delta t} \cdot \frac{C_t^{\gamma}}{\gamma} dt\right] \tag{14}$$

To solve the model we must specify the relationships between the two stochastic terms of the state equation, (12). As in Stein (2003) we assume that speculative capital gains and capital productivity are independent of one another. Hence,

$$cov\left(dW_{p,t}, dW_{b,t}\right) = 0$$

To solve (14) subject to (12) we apply the dynamic programming methodology to the stochastic optimal control problem (Chang, 2004; Fleming & Rishel, 1975). The Hamilton-Jacobi-Bellman (HJB) equation is

$$0 = \max_{u}\{U(c) - \delta V(X) + \mathcal{A}^u V(X)\} \tag{15}$$

where $\mathbf{u} \equiv \{c^*, f^*\}$, are the instantaneous values of consumption and leverage that optimize (15). As before δ is the continuous discount rate. The term $A\mathbf{u}$ is the backward operator applied to the still-undefined value function $V(X)$. Using Itō's Lemma, this can be specified as

$$\mathcal{A}^u V(X) = \frac{1}{dt} E\left[V'(X)dX + \frac{1}{2}V''(X)(dX)^2\right]$$

Our interest, however, is not the optimized path of the value function – we need only the optimal controls that the rational agent should employ. These are (see Appendix A), for consumption:

$$C^* = [A]^{\frac{1}{1-\gamma}} X$$

$$c^* = [A]^{\frac{1}{\gamma-1}} \tag{16}$$

where A is an arbitrary contant. And, the core result of the optimal choice leverage is

$$f_t^* = \frac{\mu_t + \beta - r_t}{(1-\gamma)\cdot\sigma^2} - 1 \qquad (17)$$

Where $\sigma^2 = \sigma_p^2 + \sigma_b^2$ and can be interpreted as the total asset risk. Equation (17) contains some readily apparent insights. First optimal leverage is inversely related to risk. Second, f_t^* is increasing in the expected capital gains rate, μ_t, and the average rate of productivity, β, but decreasing in the prevailing interest rate, r_t.

The optimal leverage solution in (17) is neither purely pro- nor counter-cyclical. Indeed, f_t^* is procyclical insofar as net expected returns on assets ($\mu_t + \beta - r_t$) increase during an expansion given non-increasing risk ($\Delta\sigma^2 \leq 0$).[5] This is a crucial result that sets the present model apart from typical certain-equivalent approaches. Adrian and Shin (2013) rightly criticize the extirpation of procyclical leverage models from macro-financial models. They point out that the near-universal application of concave utility functions enforces counter-cyclical leverage patterns. Although we have used such a utility function here, we have avoided the preordained, counter- cyclical leverage result because: (a) leverage is a control variable, and; (b) the SOC methodology yields time-varying optimal controls. The SOC feedback controls are "fundamentally different from the 'forward looking/certainty equivalent' models in the economics literature" (Fleming & Stein, 2004, p. 985). Indeed, as we see in section 4, f_t^* does not follow a path typical of forward-looking, rational expectations models. The result of the SOC approach is not only more a realistic time path of the control variable, but an optimal leverage estimate that can be compared directly with actual/observed leverage.

IV. Empirical Data for Constructing Optimal Leverage

Quarterly household data is taken from the Bureau of Economic Analysis (BEA), the Federal Reserve's Flow of Funds (FoF) account and Statistics Canada. These data are proxied by household assets and effective mortgage interest paid in either country. Obtaining data for household's capital gains is, however, somewhat more complicated. A perennial issue in applying optimal decision rules to financial data is separating 'fundamental changes' in price from purely speculative ones. In the present context this means $\frac{dP_t}{P_t}$ (speculative capital gains) and b_t (capital productivity) are not independently observed. Rather we are able to view only aggregate price changes. Following Stein (2012) we proxy the productivity, or "utility," of home ownership by

[5] In our present investigation procyclical leverage necessarily holds since σ2 is treated as a constant parameter throughout.

homeowners' imputed rent.[6] Statistics Canada estimates imputed rent by scaling total rent paid by the relative number of owner-occupied dwellings, housing size, a quality measure and historical ownership trends (see Statistics Canada, 2008). Dividing aggregate imputed rent by the stock of residential structures owned by households generates our proxy for the rate of housing productivity in Canada, b_t. The sample mean of b_t is the value of β. The BEA calculates the imputed rent paid by US owner-occupied households. However, the BEA reports imputed rent only as an annual total and with a delay. American owner-occupied households' 2012 imputed rent is the most recent available datum.[7] We assume payments are spread evenly over the year's quarters. This implies a dollar-valued flow of the utility of home ownership, which is then turned into a rate of return by dividing by the quarterly total stock of owner-occupied real estate, available quarterly in the FoF accounts. This stock measure, unfortunately, is for the household and NPO sector. This imperfect data set generates our quarterly b_t figure for the United States. The sample mean is β.[8]

In section 3 we argued that the expected capital gains rate, μ_t should be a positive function of total asset value increases. We operationalize this as the smoothed trend difference between observed housing index growth and its productivity rate, bt. This follows directly from the treatment of dP_t/P_t as pure speculative capital gains that, nevertheless, feed into the agent's net worth. Total price changes for Canada are taken from Teranet's Housing Price Index (HPI). The US housing price index (HPI) is taken from the original 10-city Case-Shiller index with January 2000 = 100. The rough difference between the percentage growth rate of HPI and bt is smoothed by the Hodrick-Prescott filter using a multiplier value λ = 14400. That is

$$\mu_t = HP\left(\frac{dHPI_t}{HPI_t} - b_t\right)$$

The filtered trend is then treated as the variable capital gains trend, μ_t.[9] For both countries this gives a variable but highly smoothed expected capital gains trend.

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[6] In national income accounting owner-occupied houses are treated as unincorporated firms pro- viding a service (living space) to the tenants-cum-owners.

[7] The 2012 data was released in February 2014. See BEA NIPA, Table 7.12 line 153.

[8] Note that the housing stock values used by the Fed and Statistics Canada are market values. This is appropriate since the rate of return on housing ownership would fall if, all else equal, a property's value increased. Secondly, both the US and Canadian housing stock data is taken from the household sectors' balance sheet assets meaning that there is no danger of counting commercially rented spaces.

[9] We thank Willi Semmler for suggesting this empirical approach.

Finally we must estimate $\sigma_b^2 + \sigma_p^2$. One method of estimation is to sum the sample variances of b_t and the cyclical component of the HP filtered series $\frac{dHPI_t}{HPI_t} - b_t$. However, this presumes our assumption that $\text{cov}\left(dW_{(p,t)}, dW_{b,t}\right) = 0$ is correct and places additional onus on the validity of the HP filter. We therefore rely on the (directly observed) sample variances of the housing price indices, var $\frac{(dHPI_t)}{HPI_t}$. This is admissible because b_t and dP_t/P_t are the only constitutive components of housing price changes. Moreover, if there is some correlation between $b_t\, dt$ and $dP_t/P_t\, dt$ then the denominator in (17) would contain this additional information, which would be omitted in the former estimate. That said, both approaches yield estimates on the same order of magnitude and, therefore, produce little difference in the calculation of optimal leverage.[10]

Table 1 presents the summary statistics of the variables used to construct f_t^* and the observed level of household leverage for Canada and the United States. Note that Canadian housing price data is available only from 1999 onwards, hence there are only 56 observations for $\frac{dHPI_t}{HPI_t}$ and μ_t. Data for imputed rent, housing stock and paid interest needed to construct b_t and r_t are available from 1990 (no. 96). Table 1 therefore presents more information than can be used in the construction of Canada's f_t^*, for which all data is take from 2000 Q1 onwards.[11] In constructing f_t^* for Canada all series begin with 2000 Q1. Housing price data also limits the US series, but, as the 10-city Case-Shiller index is available from 1987, all US figures in Table 1 are already circumscribed to start in 1988 Q1.[12]

[10] For both Canadian and US data the variance of housing price growth that we use is greater than the summation approach. For Canada this is $\sigma_b^2 + \sigma_p^2 = 0.00759$ versus 00.409. This suggests that $\text{cov}\left(dW_{(p,t)}, dW_{b,t}\right) = 0$ does not strictly hold.

[11] Teranet's 11-city housing price index begins in 1999 Q1, meaning year-on-year changes can begin only in the second year of the available data.

[12] See footnote 11.

Table 1: Summary Statistics for Data Series

Variable	No. Obs.	Arithmetic Mean	Variance	Minimum	Maximum
Canada					
f_t	96	0.2094	0.0004	0.1807	0.2549
$\frac{dHPI_t}{HPI_t}$	56	0.0610	0.0015	-0.0602	0.1316
b_t	96	0.0255	0.0000092	0.0209	0.0301
r_t	96	0.0176	0.0000425	0.0087	0.0308
μ_t	56	0.0375	0.0001	0.0110	0.0510
United States					
f_t	104	0.1850	0.0006	0.1546	0.2537
$\frac{dHPI_t}{HPI_t}$	104	0.0364	0.0076	0.1546	0.2537
b_t	100	0.0163	0.0000039	0.012	0.0196
r_t	100	0.0161	0.0000181	0.0094	0.0245
μ_t	100	0.0169	0.0019	-0.0744	0.0871

V. Interpreting Optimal Leverage and the Early Warning Sign

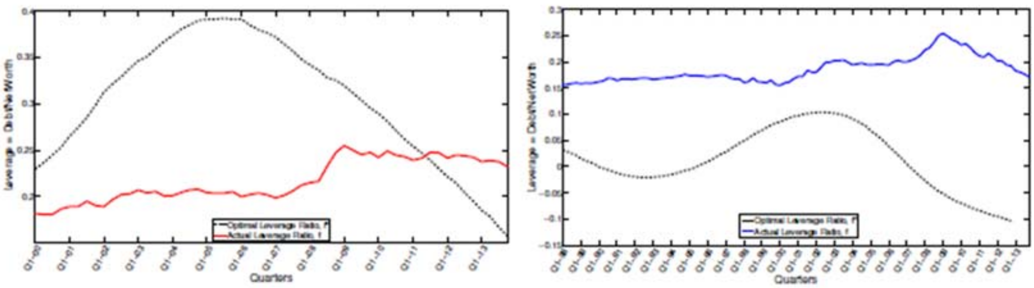
Figure 3 plots the optimal and actual leverage ratios for Canada (Fig. 3a) and the United States (Fig. 3b). The optimal leverage ratio is computed by equation (17) using the empirical estimates described in section 4. As previously noted, the low variance measures produce raw f_t^* ratios that are two orders of magnitude too large; we therefore scale both countries' ratios down by 10^2. Even with this scaling, the two figures show that our constructed measure of f_t^* is not robust. Using the level estimate of f_t^* produces an EWS measure as a difference of ratios for which there is no singular, let alone correct, interpretation of the magnitude of Ψ_t. Even if our optimal leverage measures were robust, the level measure of the EWS would still face interpretative ambiguity. Therefore, we must follow Stein's approach and compare the normalized values of f_t and f_t^*. Normalization centers the data at zero and, more importantly, eliminates the estimated parameters (*e.g.*, σ^2) from the optimal leverage estimate.

In Figure 4 all US variables are normalized by their pre-crisis norms. Specifically,

$$N(f_t^*) = \frac{f_t^* - \overline{f^*}}{\sigma_{f^*}} \qquad (18)$$

$$N(f_t^*) = \frac{f_t - \bar{f}}{\sigma_f} \qquad (19)$$

Figure 3: Optimal versus Actual Household Leverage

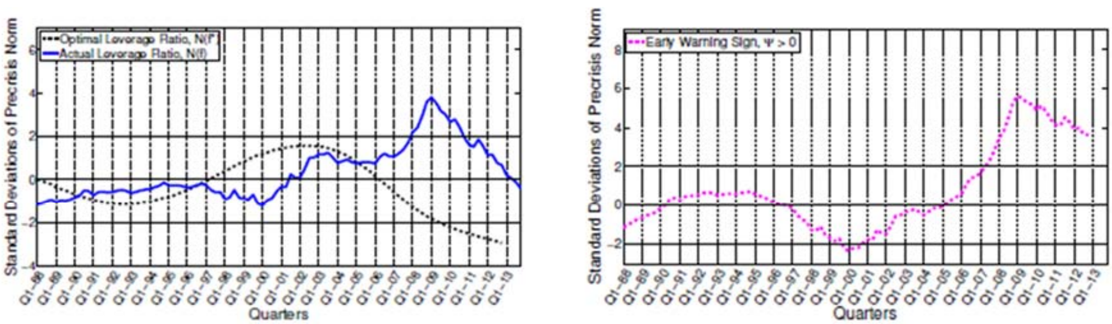


(a) Household Leverage in Canada (b) Household Leverage in the United States

Author's calculations. Risk aversion parameter are γ = 0.9 for Canada and the United States. See Table B.2 for details.

where the $\bar{f}$ and σ_f are the sample mean and standard deviation through 2008 Q1. This cut point is chosen since fragility had, in retrospect, clearly built up to very high levels by early 2008, but the crashes of Bear Stearns in March and Lehman Brothers in September could not have yet spilled over into the housing sector. That is, we have normalized US leverage ratios according to their 'crash-free' histories.

Figure 4: United States Normalized Leverage and EWS

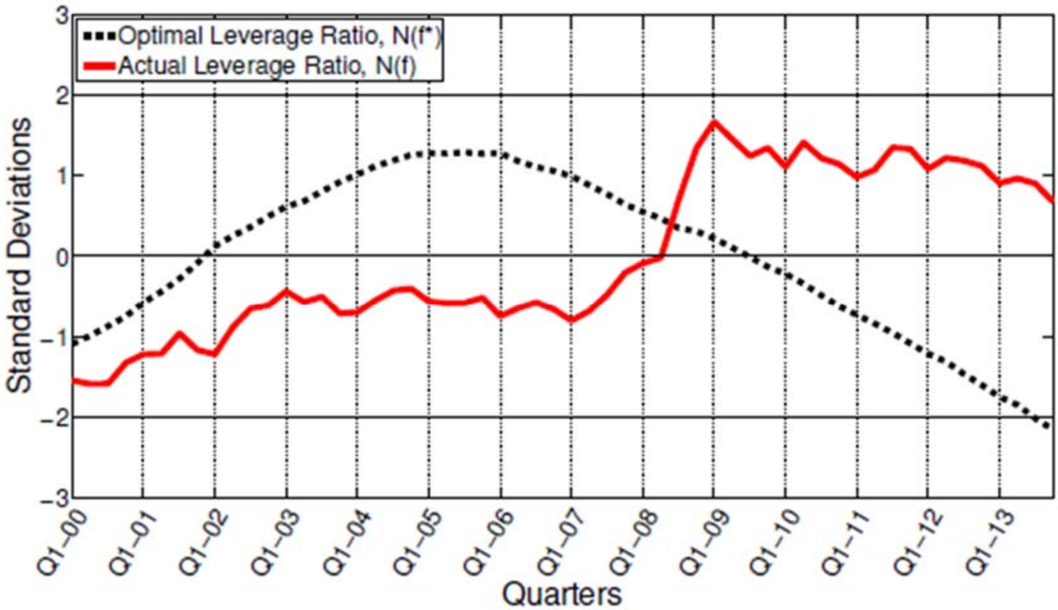


(a) Normalized Leverage Rates (b) Early Warning Sign, $N(f_t) - N(f_t^*)$

Author's calculations. See Table B.2 for details.

Figure 4a plots the series produced by (18) and (19) for the United States. As already shown, US household leverage reached unprecedented heights in 2009 Q1, from which it has rapidly fallen to its pre-crisis average. Further, it is now clear that the (normalized) optimal leverage measure began a steady decline several years prior to the jump in observed leverage. Indeed, since 2006 Q1 American households' optimal leverage has been below its historic, pre-crisis mean. More tellingly, the US optimal leverage figure fell very rapidly, decreasing by 3 standard deviations over four years (from approximately +1.5 2004 Q1 to -1.5 in 2007 Q4). Falling optimal leverage has, of course, limited the retrenchment of the US EWS. In Figure 4b $N(\Psi_t)$ also peaks in 2009 Q1, at +5.6 standard deviations above the pre-crisis average, but has since fallen to only +3.7 s.d. by 2012 Q4 – still well above historic norms. This normalized EWS indicates, we think correctly, that US households continue to be in a precarious financial state that is only slowly improving. Yet, the most important result from the $N(\Psi_t)$ series is that it was an early indicator of financial fragility. The EWS was 2 standard deviations above its historical norm in the first half of 2007, before any widespread acknowledgement of financial fragility, let alone an incipient crisis. Thus, with data available by the beginning of 2008, the normalized US early warning sign clearly evidences a high degree of stress as of 2007 Q2, if not sooner. If applied contemporaneously to the US housing crisis, the normalized EWS would have provided an unambiguous interpretation of financial fragility in the household sector.

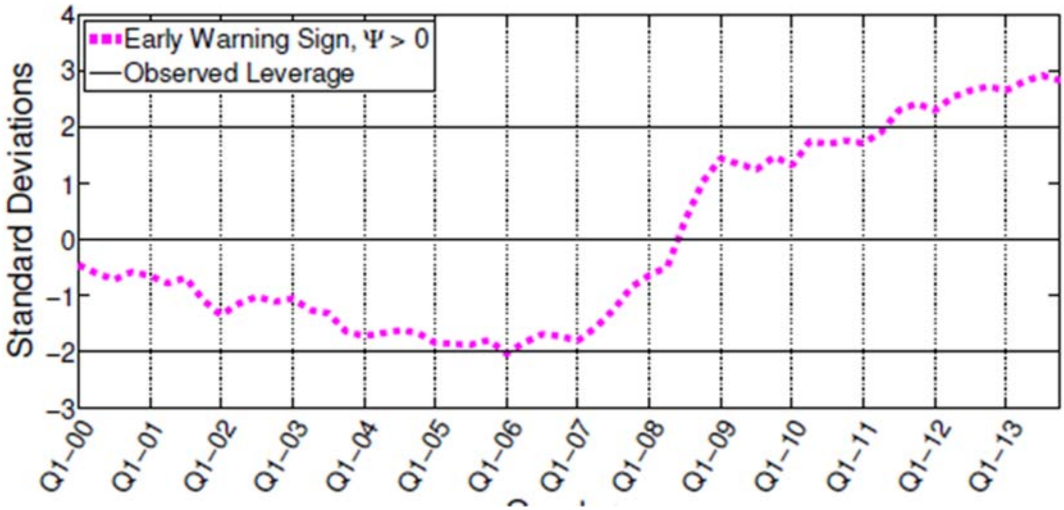
Figure 5: Canada's Normalized Actual and Optimal Leverage



Author's calculations. See Table B.2 for details.

As Canadian households have not experienced a crisis we use the entire data set (*i.e.*, through 2013 Q4) to normalize actual and optimal leverage by (18) and (19). The normalize leverage series and EWS data for Canada are plotted, respectively, in Figures 5 and 6. Although Canadian household leverage hit the same level as in the US (≈ 25%), Figure 5 includes this maximum point in its normalization. Therefore, $N(f_t)$ peaks at +1.67 standard deviations in 2009 Q1 versus the contemporaneous +6.3 s.d. in the US. Even if one normalizes US leverage over the entire data series this maximum is still an incredible +2.77 standard deviations above the mean. Rather than mimicking the US 2008 boom/bust cycle, the normalized Canadian leverage series more closely aligns with the US $N(f_t)$ data from 2000 to 2007. During this period US household leverage jumped from a low point in 2000 Q1 to approximately +1.5 s.d. above its average around which it hovered for the next 5 years. The Canadian $N(f_t)$ similarly jumped in 2008 from a lower level, and has been hovering between +1 and +1.5 standard deviations. However, if the downward trend since 2012 Q4 continues then Canadian households may safely reduce their exposure.

Figure 6: Canada's Early Warning Sign



Author's calculations. See Table B.2 for details.

However, the second similarity to the pre-crisis US data is the steep decline in the optimal leverage measure. Canada's fall in optimal leverage has been somewhat less rapid than in the US. From 2009 Q1 to 2013 Q4 optimal leverage fell by over 2 standard deviations, from +0.22 to -2.15. Since the decline in optimal leverage continues unabated, the Canadian EWS has continued to rise even as actual leverage has begun to decline (see Fig. 6). The resulting EWS pattern for Canada is quite distinct from that of the United States. After an initial rapid build during 2008 to 1.44 s.d. in 2009 Q1, the Canadian EWS remained fairly constant through to 2010 Q1 at which point it began to rise again. The $N(\Psi_t)$ peaked at 2.92 s.d. in 2013 Q3 before scaling back slightly to 2.83 in the next quarter (which is nevertheless the second highest EWS measure for Canada). Given the success of the normalized EWS in leading the US housing crisis, the the persistent elevation in Canada's EWS does not bode well for the country's households.

VI. Implications and Further Research

After developing and amending Stein's optimal leverage model, we applied its Early Warning Sign (EWS) to the completed US housing boom-bust cycle of the 2000s and to the still booming Canadian housing market. Level estimates of the EWS were not robust and so we followed Stein's method of normalizing the observed and optimal leverage measures. The optimal leverage trends for both countries (normalized by the pre-boom sample data) revealed some similar patterns. For many years US and Canadian households evidenced only small deviations from their historic EWS averages only to be followed by a sustained rises above the norms.[13] In the US, the elevated EWS, which began in 2002, rapidly rose over 2006-2007, and was then followed by the infamous crash in 2008 (see Fig. 4b). In Canada, the elevated EWS has been sustained since 2011 and has been slowly creeping up through 2013. From this level any retrenchment would be far less than that experienced in the US, but it would nevertheless represent a serious correction. Our conclusion, therefore, is that Canadian households continue to be in a precarious financial position.

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[13] Of course, this sustained elevation was lower for the US (≈ 1 s.d.) than for Canada (≈ 2 s.d.).

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Appendix A Stochastic Optimal Control Solution

The balance sheet identity for the flow of assets, liabilities and net worth is

$$dX_t = dK_t - dL_t \tag{A.1}$$

The total value of assets (capital) grows according to

$$dK_t = I_t dt + \frac{K_t(dP_t)}{P_t} \tag{A.2}$$

and liabilities according to

$$dL_t = (r_t L_t + C_t + I_t)dt - Y_t dt \tag{A.3}$$

where capital generates value-added output Y_t, *by*

$$Y_t dt \equiv b_t K_t dt \tag{A.4}$$

Capital's productivity, b_t, is a random process with mean $\bar{b}$. It is defined by

$$b_t dt = \beta dt + \sigma_p dW_p \tag{A.5}$$

Speculative price changes grow according to geometric Brownian motion

$$\frac{dP_t}{P_t} = \mu_t dt + \sigma_p dW_p \tag{A.6}$$

To solve for the optimal control functions c_t^* and f_t^* using the above equations first substitute (A.6) into (A.2) and (A.5) into (A.3), and then equations (A.2) and (A.3) into (A.1). Thus,

$$dX_t = \frac{K_t(dP_t)}{P_t} + K_t b_t dt - L_t r_t dt - C_t dt$$

Equivalently

$$\frac{dX_t}{X_t} = (1+f_t)\left(\mu_t dt + \sigma_p dW_p\right) + (1+f_t)(\beta dt + \sigma_b dW_b)dt - f_t r_t dt - c_t dt$$

Rearranging this becomes

$$dX_t = X_t\left\{[(1+f_t)(\mu_t + \beta) - f_t r_t - c_t]dt + (1+f_t)(\sigma_b dW_b + \sigma_p dW_p)\right\} \tag{A.7}$$

where $c_t \equiv \frac{C_t}{X_t}$, $f_t \equiv \frac{L_t}{X_t}$ and $1 + f_t = \frac{A_t}{X_t}$.

Now setup the objective function as a continuous-time HARA function

$$V(X) = \max_{c_t, f_t} E_0 \left[\int_0^\infty e^{-\delta t} \frac{1}{\gamma}(C_t)^\gamma dt\right] \tag{A.8}$$

Which is subject to (A.7). The dynamic programming equation uses Itō's Lemma in the continuous-time Bellman equation which has the generic form

$$0 = \max_u\{U(c) - \delta V(X) + \mathcal{A}^u V(X)\} \tag{A.9}$$

where $u \equiv \{c, f\}$, are the instantaneous values of consumption and leverage chosen to optimize (A.9). $\mathcal{A}^u$ is the backward stochastic operator applied to the undefined value function $V(X)$. For an autonomous, scalar diffusions, *i.e.* (A.7), this is defined as (Fleming & Rishel, 1975, Chapters V and VI).

$$\mathcal{A}^u(s) = \frac{\partial}{\partial y} f(s, y, v) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\sigma^2(s, y, v) \tag{A.10}$$

where y is the state variable, v is the control, and s is the initial time $s < t$, and the SDE has the generic form $dy = f(s,y,v)dt + \sigma(s,y,v)dW_t$.

Applied to our particular value function the backward operator may be written more explicitly as

$$\mathcal{A}^u V(X) = \frac{1}{dt} E\left[\frac{\partial V(X)}{\partial X} dX + \frac{1}{2}\frac{\partial^2 V(X)}{\partial X^2}(dX)^2\right]$$
$$= V_X' \cdot X[(1+f_s)(\mu_s+\beta) - f_s - c_s] + V_{XX}'' \frac{1}{2} X^2\left[(1+f_s)^2(\sigma_p^2+\sigma_b^2)\right]$$

Substituting $\mathcal{A}^u V(X)$ and $U(C) = \frac{1}{\gamma}C^\gamma$ into (A.9) gives

$$\delta V(X) = \max_{f,c}\left\{\frac{1}{\gamma}C^\gamma + V_X' \cdot X[(1+f_s)(\mu_s+\beta) - f_s - c_s] + V_{XX}'' \frac{1}{2} X^2\left[(1+f_s)^2(\sigma_p^2+\sigma_b^2)\right]\right\} \quad (A.11)$$

where *δV(X)* has been removed from the maximum operator because it is independent of controls. Clearly equation (A.11) is a second-order, nonlinear ordinary differential equation. Therefore, by the method of undetermined coefficients we may suppose that $V(X) = \frac{A}{\gamma} \cdot X^\gamma$ solves the ODE given the arbitrary constant *A*. Then,

$$V_X' = AX^{\gamma-1} \quad (A.12)$$

$$V_{XX}' = A(\gamma-1)X^{\gamma-2} \quad (A.13)$$

Solving for the intertemporal maximum C^* by the first-order condition of the DP maximization

$$0 = C^{\gamma-1} - \overbrace{AX^{\gamma-1}}^{V_X'} \cdot \frac{\overbrace{\partial(X\cdot c)}^{=C}}{\partial C}$$

$$(C^*)^{\gamma-1} = AX^{\gamma-1}$$

$$C^* = [A]^{\frac{1}{\gamma-1}}X \quad (A.15)$$

$$c^* = A^{-\frac{1}{1-\gamma}} \quad (A.16)$$

Similarly to solve for f^* we find

$$arg\max_f\left\{V_X' \cdot X[(1+f)(\mu_s+\beta) - fr_s - c] + V_{XX}'' \frac{1}{2} X^2\left[(1+f)^2(\sigma_p^2+\sigma_b^2)\right]\right\}$$

Now differentiate with respect to f_s and set equal to zero:

$$0 = AX^{\gamma}(\mu_s + \beta - r_s) - A(1-\gamma)X^{\gamma}\big((1+f)(\sigma_p^2 + \sigma_b^2)\big)$$

$$(1-\gamma)\left(\sigma_p^2 + \sigma_b^2\right)f^* = \mu_s + \beta - r_s - (1-\gamma)\left(\sigma_p^2 + \sigma_b^2\right)$$

$$f^* = \frac{\mu_s + \beta - r_s}{(1-\gamma)\cdot\left(\sigma_p^2 + \sigma_b^2\right)} - 1$$

□

Appendix B List of Variables

Table B.1: Optimal Leverage Model Variables

Name	Variable	Notes
X_t	Net Worth	Book value of equity, defined by balance sheet identity $K_t = X_t + L_t$. State variable.
L_t	Debt	Total stock of loans held by the agent.
N_t	Capital Stock	Physical amount of capital.
Pt	Capital Price	Market value of capital (proportional changes are capital gains).
K_t	Total Assets	Capital made up of debt and equity. Also the monetary value of physical capital: $K_t = P_t N_t$.
$f_t = \frac{L_t}{X_t}$	Leverage Ratio	Control variable. Capital/Equity ratio, $\frac{K_t}{X_t} = 1 + f_t$.
$c_t = \frac{C_t}{X_t}$	Consumption Ratio	Control Variable. Ratio of consumption to net worth.
$\frac{dP_t}{P_t}$	Speculative Capital Gains	Geometric Brownian motion with variable drift, $\frac{dP_t}{P_t} = \mu_t dt + \sigma_p dW_t$.
r_t	Interest Rate	Ornstein-Uhlenbeck mean-reverting stochastic process, $dr_t = \theta(\bar{r} - r_t)dt + \sigma_r dW_r$ (the Vasicek model).
b_t	Capital Productivity	Stochastic non-differential process, $b_t = \beta dt + \sigma_b dW_b$.
Y_t	Net Value Added	Akin to GDP or total output, $Y_t = b_t K_t$ where the productivity term is a stochastic process.

Table B.2: Parameter Values for Optimal Leverage Estimate

Variable	Description	Value	
		Canada	USA
Calibrated Parameters			
β	Mean productivity rate	0.0235	0.0163
$\bar{\mu}$	Mean expected speculative capital gains	0.0375	0.0169
$\bar{r}$	Mean effective mortgage interest rate	0.0129	0.0161
σ_b^2	Productivity diffusion coefficient	0.0000055	0.0000039
σ_p^2	Capital gains diffusion coefficient (variance of cyclical trend from HP filter on $dHPI_t/HPI_t - b_t$)	0.0012	0.0041
σ_r^2	Interest rate diffusion coefficient	0.0000077	0.0000138
γ	Risk aversion (given)	0.9	0.9
σ^2	Total risk value var($dHPI_t/HPI_t$)	0.0015	0.0072
$\overline{f^*}$	Mean optimal leverage	0.3043	0.0137